

An Algorithm for Calculating Terms of a Number Sequence using an Auxiliary Sequence

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Abstract

Number sequences defined by a linear recursion relation are studied by means of generating functions. Indices of the terms in the recursion relation have arbitrary differences. In addition to formulas for the n th term an algorithm is derived for calculating the n th term even without an expression in closed form.¹

The numbers are denoted $a_n, n = 0, 1, 2, \dots$ and are defined by initial values

$$a_0 = \alpha_0, a_1 = \alpha_1, \dots a_{N-1} = \alpha_{N-1} \text{ fr } N \geq 2$$

and a recursion relation

$$a_n = a_{n-1} + a_{n-N} \text{ for } n = N, N+1, \dots$$

The number N has great influence on the complexity of derivations and calculations since it gives the degree of the polynomial in the denominator of the generating function.

1 Generating function

For generalities on using generating functions, see [1], p 337 f and [2], p 445 f.

¹This is version 2. An essential addition has been made at the beginning of page 3.

A generating function for the number sequence $(a_n)_{n=0}^{\infty}$ is $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Manipulation of indices and use of the recursion relation and the initial values yield

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} a_{n+N} x^{n+N} + \sum_{n=0}^{N-1} \alpha_n x^n \\
&= \sum_{n=0}^{\infty} (a_{n+N-1} + a_n) x^{n+N} + \sum_{n=0}^{N-1} \alpha_n x^n \\
&= x \sum_{n=0}^{\infty} a_{n+N-1} x^{n+N-1} + x^N \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{N-1} \alpha_n x^n \\
&= x \sum_{n=0}^{\infty} a_n x^n - x \sum_{n=0}^{N-2} a_n x^n + x^N \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{N-1} \alpha_n x^n \\
&= (x + x^N) f(x) - \sum_{n=1}^{N-1} \alpha_{n-1} x^n + \sum_{n=1}^{N-1} \alpha_n x^n + \alpha_0
\end{aligned}$$

from which we get

$$f(x) = \frac{\sum_{n=0}^{N-1} (\alpha_n - \alpha_{n-1}) x^n}{1 - x - x^N} . \quad (1)$$

Here we have defined $\alpha_{-1} = 0$ in order to be able to write sums more concisely.

Since $a_n = f^{(n)}(0)/n!$ (taylor expansion) this gives

$$a_n = \sum_{\ell=0}^{N-1} (\alpha_{\ell} - \alpha_{\ell-1}) C_{n\ell} \quad (2)$$

where

$$C_{n\ell} = \frac{1}{n!} \frac{d^n}{dx^n} \frac{x^{\ell}}{1 - x - x^N} \Big|_{x=0} . \quad (3)$$

It will be seen later (section 3) that the right side expression depends on the difference $n - \ell$ only so $C_{n\ell} = D_{n-\ell}$ where D_n is a certain function of n .

Now, let δ denote the positive, real zero² of the polynomial $x^N + x - 1$; $0 < \delta < 1$.

²For odd N this is *the only* real zero and it is always in the interval $]0, 1[$.

We now define another number sequence $(\kappa_{n,\ell}^N)_{\ell=0}^{N-1}$ by

$$(\delta^{N-1} + 1)^n = \sum_{\ell=0}^{N-1} \kappa_{n,\ell}^N \delta^\ell$$

where N and n are parameters specifying the number sequence and ℓ index, numbering the terms of the sequence.

The exponents of δ in the expansion of the left hand side can be reduced to at most $N - 1$ since $\delta^N + \delta - 1 = 0$, i.e. $\delta^N = 1 - \delta$. The powers of δ with indices $0, 1, \dots, N - 1$ are linearly independent as follows from the fact that the polynomial $x^N - x - 1$ is irreducible over the integers [3], implying that this is the minimal polynomial of δ . This motivates the identification of coefficients in the following.

$$\begin{aligned} (\delta^{N-1} + 1)^{n+1} &= \sum_{\ell=0}^{N-1} \kappa_{n+1,\ell}^N \delta^\ell \\ &= \sum_{\ell=0}^{N-1} \kappa_{n,\ell}^N \delta^{N-1+\ell} + \sum_{\ell=0}^{N-1} \kappa_{n,\ell}^N \delta^\ell \\ &= \kappa_{n,0}^N \delta^{N-1} + \kappa_{n,1}^N + \kappa_{n,0}^N + \sum_{\ell=1}^{N-2} \kappa_{n,\ell+1}^N \delta^\ell \end{aligned}$$

giving

$$\begin{aligned} \kappa_{n+1,N-1}^N &= \kappa_{n,0}^N \\ \kappa_{n+1,\ell}^N &= \kappa_{n,\ell+1}^N \quad (\ell = 1, 2, \dots, N - 2; N \geq 3) \\ \kappa_{n+1,0}^N &= \kappa_{n,0}^N + \kappa_{n,1}^N \end{aligned}$$

and then

$$\begin{aligned} \kappa_{n+1,N-1}^N &= \kappa_{n,0}^N = \kappa_{n+1,0}^N - \kappa_{n,1}^N \\ &= \kappa_{n+2,N-1}^N - \kappa_{n-1,2}^N \\ &= \kappa_{n+2,N-1}^N - \kappa_{n-2,3}^N = \dots = \kappa_{n+2,N-1}^N - \kappa_{n-N+2,N-1}^N \end{aligned}$$

which finally gives

$$\kappa_n = \kappa_{n-1} + \kappa_{n-N} \quad . \quad (4)$$

Here we have excluded upper index and the second lower index since these have the same value in each term in each separate case (i.e. for each polynomial $x^N + x - 1$ and each exponent n). Hence we have defined a new number sequence $(\kappa_n)_{n=0}^\infty$ where now n is an index, numbering the terms of the sequence. It is easily seen that (4) holds also for $N = 2$.

Since (4) is the same recursion relation as the one for a_n we get by means of (2)

$$\kappa_n = \kappa_0 D_n + \sum_{\ell=1}^{N-1} (\kappa_\ell - \kappa_{\ell-1}) D_{n-\ell} \quad (5)$$

and

$$a_n = \alpha_0 D_n + \sum_{\ell=1}^{N-1} (\alpha_\ell - \alpha_{\ell-1}) D_{n-\ell} . \quad (6)$$

Independently of number sequence a_n ³

$$\kappa_0 = 0, \quad \kappa_n = 1 \text{ for } 0 < n \leq N-1$$

($N \geq 2$) which implies that (5) can be written

$$\kappa_n = \sum_{\ell=1}^{N-1} (\kappa_\ell - \kappa_{\ell-1}) D_{n-\ell} = D_{n-1} . \quad (7)$$

Finally, substituting (7) into (6) gives

$$a_n = \alpha_0 \kappa_{n+1} + \sum_{\ell=1}^{N-1} (\alpha_\ell - \alpha_{\ell-1}) \kappa_{n+1-\ell} . \quad (8)$$

In specific cases (given N, α_ℓ) this gives a relation between a_n and $\kappa_n = \kappa_{n, N-1}^N$ by which a_n can be calculated by expanding $(\delta^{N-1} + 1)^n$ and noticing the coefficient for δ^{N-1} , i.e. $\kappa_{n, N-1}^N$. Although this does not mean that we have an expression for a_n in closed form, for specific, even large, values of n the expansion often gives an easier calculation of a_n than using a closed formula.

Notice that the exact expression for D_n is not needed although, of course, it gives a_n as an explicit function of n by (6).

³The proof of this is somewhat complicated and is given in an appendix. In specific cases with small n it is easily shown. For instance $(\delta^{N-1} + 1)^2 = \delta^{2N-2} + 2\delta^{N-1} + 1 = \delta^N \delta^{N-2} + 2\delta^{N-1} + 1 = (1 - \delta)\delta^{N-2} + 2\delta^{N-1} + 1 = \delta^{N-1} + \delta^{N-2} + 1$ so $\kappa_2 = 1$ and obviously $\kappa_0 = 0$ and $\kappa_1 = 1$ (for all N).

2 Examples

Some useful relations to reduce polynomials in δ to polynomials with exponents in the interval $[0, N-1]$ and to increase/decrease the exponent n in the expression of the form $(\delta^{N-1} + 1)^n$:

$$\delta^N = 1 - \delta, \quad \frac{1}{\delta} = \delta^{N-1} + 1 .$$

Ex 1: $N = 2$ and $\alpha = 0, 1$ giving Fibonacci's number sequence.

In this case $\delta^2 + \delta - 1 = 0$, i e $\delta^2 = 1 - \delta$ and $1/\delta = \delta + 1$. The terms of the sequence are denoted F_n and equation (8) gives $F_n = \kappa_n$. We will find F_{40} .

$$(\delta + 1)^2 = 1 - \delta + 2\delta + 1 = \delta + 2 ,$$

$$(\delta + 1)^4 = (\delta + 2)^2 = 1 - \delta + 4\delta + 4 = 3\delta + 5$$

and similarly for $(\delta + 1)^8$, $(\delta + 1)^{16}$ och $(\delta + 1)^{32}$. By means of this we get

$$\begin{aligned} (\delta + 1)^{40} &= (\delta + 1)^{32} \cdot (\delta + 1)^8 = (2178309\delta + 3524578)(21\delta + 34) \\ &= 45744489(1 - \delta) + (74062506 + 74016138)\delta + 3524578 \cdot 34 \\ &= 102334155\delta + (\dots) . \end{aligned}$$

Thus, $F_{40} = 102334155$.

Ex 2: $N = 3$ and $\alpha = 0, 1, 2$.

In this case $\delta^3 + \delta - 1 = 0$, i e $\delta^3 = 1 - \delta$, $\delta^4 = \delta - \delta^2$ and $1/\delta = \delta^2 + 1$. Equation (8) gives $a_n = \kappa_n + \kappa_{n-1}$. We will find a_{16} .

$$(\delta^2 + 1)^2 = \delta - \delta^2 + 2\delta^2 + 1 = \delta^2 + \delta + 1 ,$$

$$(\delta^2 + 1)^4 = (\delta^2 + \delta + 1)^2 = \delta - \delta^2 + \delta^2 + 1 + 2 - 2\delta + 2\delta^2 + 2\delta = 2\delta^2 + \delta + 3$$

and similarly for $(\delta^2 + 1)^8$ and $(\delta^2 + 1)^{16}$. This gives

$$(\delta^2 + 1)^{15} = (\delta^2 + 1)^{16} \cdot \delta = (189\delta^2 + 129\delta + 277)\delta = 129\delta^2 + (\dots)$$

and, thus, we get

$$a_{16} = 189 + 129 = 318 .$$

Ex 3: $N = 3$ and $\alpha = 1, 2, 3$ and the recursion relation $a_{n+3} = a_{n+2} \cdot a_n$.

The number sequence⁴ is, thus, 1, 2, 3, 3, 6, 18, 54, 324, ...

Let $b_n = \ln a_n$. Then $(b_n)_{n=0}^\infty$ is a number sequence with a linear recursion relation $b_{n+3} = b_{n+2} + b_n$ and can be treated like in ex 2 with initial values $\beta = \ln \alpha$, so 0, $\ln 2$, $\ln 3$ and $N = 3$. We will find a_{16} .

$$b_n = \ln 2 \cdot \kappa_n + (\ln 3 - \ln 2) \cdot \kappa_{n-1} = (\kappa_n - \kappa_{n-1}) \ln 2 + \kappa_{n-1} \ln 3 ,$$

$$a_n = 2^{\kappa_n - \kappa_{n-1}} \cdot 3^{\kappa_{n-1}}$$

so we need κ_{16} and κ_{15} , which have already been found in ex 2, $\kappa_{16} = 189$ and $\kappa_{15} = 129$. Thus,

$$a_{16} = 2^{\kappa_{16} - \kappa_{15}} \cdot 3^{\kappa_{15}} = 2^{189-129} \cdot 3^{129}$$

$$= 2^{60} \cdot 3^{129}$$

$$= 40779472028876430259264292468803306803871352789421825624677506478583962620919808$$

(80 digits).

Ex 4: $N = 4$ and $\alpha = 0, 1, 2, 3$.

$$a_n = \kappa_n + \kappa_{n-1} + \kappa_{n-2}$$

Ex 5: $N = 4$ and $\alpha = 1, 1, 1, 1$.

$$a_n = \kappa_{n+1}$$

Ex 6: $N = 5$ and $\alpha = 0, 1, 2, 1, 1$.

$$a_n = \kappa_n + \kappa_{n-1} - \kappa_{n-2}$$

⁴This example results from a discussion of which the next number should be in a sequence starting with 1, 2, 3, 3. The idea was that you can always find a "rule" giving any next number, for instance by adaption of a polynomial. One of my former colleagues at Kunskapsgymnasiet in Gothenburg Sweden, Maria Nars, came up with the recursion relation in this example. I wanted to find a general formula and from that this article grew.

3 Formulas for D_n and a_n

Partial fractions expansion of $f(x)$ (the zeroes x_k of the denominator are simple):

$$\frac{1}{1-x-x^N} = \sum_{k=1}^N \frac{r_k}{x-x_k}$$

where

$$r_k = -\frac{1}{1+Nx_k^{N-1}}$$

so, according to (1),

$$\begin{aligned} f(x) &= \sum_{n=0}^{N-1} (\alpha_n - \alpha_{n-1}) x^n \cdot \sum_{k=1}^N \frac{r_k}{x-x_k} \\ &= \sum_{n=0}^{N-1} (\alpha_n - \alpha_{n-1}) x^n \cdot \sum_{k=1}^N \frac{-x_k}{(N-(N-1)x_k)(x-x_k)} \quad . \end{aligned}$$

Since

$$\left. \frac{d^m}{dx^m} x^\ell \right|_{x=0} = m! \delta_{m\ell}$$

and

$$\left. \frac{d^m}{dx^m} \frac{1}{x-x_k} \right|_{x=0} = \frac{(-1)^m \cdot m!}{(-x_k)^{m+1}} = -\frac{m!}{x_k^{m+1}} \quad ,$$

by Leibniz' formula for the n th derivative of a product

$$\begin{aligned} \left. \frac{d^n}{dx^n} \frac{x^\ell}{x-x_k} \right|_{x=0} &= \sum_{j=0}^n \binom{n}{j} \frac{d^j}{dx^j} x^\ell \cdot \left. \frac{d^{n-j}}{dx^{n-j}} (x-x_k)^{-1} \right|_{x=0} \\ &= -\binom{n}{\ell} \cdot \ell! \cdot (n-\ell)! x_k^{-1-n+\ell} \\ &= -n! \cdot x_k^{-1-n+\ell} \quad . \end{aligned}$$

Since

$$f(x) = \sum_{\ell=0}^{N-1} \sum_{k=1}^N \frac{(\alpha_\ell - \alpha_{\ell-1})(-x_k)}{N-(N-1)x_k} \cdot \frac{x^\ell}{x-x_k}$$

we get

$$f^{(n)}(0) = \sum_{\ell=0}^{N-1} \sum_{k=1}^N \frac{(\alpha_\ell - \alpha_{\ell-1})x_k}{N - (N-1)x_k} \cdot n! \cdot x_k^{-1-n+\ell}.$$

According to (3)

$$\begin{aligned} C_{n\ell} &= \frac{1}{n!} \cdot \frac{d^n}{dx^n} \sum_{k=1}^N \frac{r_k x^\ell}{x - x_k} \Big|_{x=0} \\ &= \frac{1}{n!} \cdot \sum_{k=1}^N r_k (-n!) x_k^{-1-n+\ell} \\ &= \sum_{k=1}^N \frac{1}{1 + N x_k^{N-1}} \cdot x_k^{-1-n+\ell} \\ &= \sum_{k=1}^N \frac{x_k^{\ell-n}}{x_k + N x_k^N} \\ &= \sum_{k=1}^N \frac{x_k^\ell}{[N - (N-1)x_k] x_k^n}. \end{aligned}$$

Obviously $C_{n\ell}$ is a function of $n - \ell$ and can be written $C_{n\ell} = D_{n-\ell}$ where

$$D_n = \sum_{k=1}^N \frac{1}{[N - (N-1)x_k] x_k^n} \quad (9)$$

and by (2), (3)

$$a_n = \sum_{k=1}^N \frac{\alpha_0 + \sum_{\ell=1}^{N-1} (\alpha_\ell - \alpha_{\ell-1}) x_k^\ell}{[N - (N-1)x_k] x_k^n}. \quad (10)$$

Hence we have obtained explicit formulas for D_n and a_n .

4 Explicit function for $N = 2$, Fibonacci again

The polynomial $x^2 + x - 1$ has the zeroes $x_1 = (-1 + \sqrt{5})/2$, $x_2 = (-1 - \sqrt{5})/2$ and $x_1 + x_2 = -1$, $x_1 \cdot x_2 = -1$. $\alpha_0 = 0$, $\alpha_1 = 1$ and (10) give

$$\begin{aligned}
F_n &= \sum_{k=1}^2 \frac{x_k}{(2-x_k)x_k^n} \\
&= \frac{1}{5(-1)^n} [x_1^n + x_2^n + 2(x_1^n x_2 + x_1 x_2^n)] \\
&= \frac{(-1)^n}{\sqrt{5}} (x_2^n - x_1^n)
\end{aligned}$$

after which substitution of x_1 och x_2 gives (Binet's formula)

$$F_n = \frac{1}{\sqrt{5}} \left\{ \left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right\} .$$

5 Explicit function for $N = 3$

Let

$$p_n \equiv (x_1 x_2)^n + (x_2 x_3)^n + (x_3 x_1)^n .$$

Then

$$p_n = 2(-\sqrt{\delta})^n \cdot T_n \left(\frac{\delta\sqrt{\delta}}{2} \right) + (\delta^2 + 1)^n$$

where T_n , for $n = 0, 1, \dots$ denotes a Chebyshev polynomial⁵

Proof for this:

The zeroes of the polynomials $x^3 + x - 1$ satisfy

$$\begin{cases} x_1 + x_2 + x_3 &= 0 , \\ x_1 x_2 + x_2 x_3 + x_3 x_1 &= 1 , \\ x_1 x_2 x_3 &= 1 . \end{cases}$$

The real zero is denoted $x_1 = \delta$ and the non-real ones can be written $x_2 = \rho e^{i\varphi}$ and $x_3 = \rho e^{-i\varphi}$. Since $x_1 x_2 x_3 = 1$, $\rho = 1/\sqrt{\delta}$ and

⁵A Chebyshev polynomial T_n is defined by $T_n(\cos \varphi) = \cos(n\varphi)$ or $T_n(x) = \cos(n \arccos x)$. The polynomial can be determined by means of trigonometric formulas for $\cos(n\varphi)$. For $n = 0 - 5$ the polynomials are $T_0(x) = 1$, $T_1(x) = x$, $T_2(x) = 2x^2 - 1$, $T_3(x) = 4x^3 - 3x$, $T_4(x) = 8x^4 - 8x^2 + 1$, $T_5(x) = 16x^5 - 20x^3 + 5x$. From the definition it follows that $|T_n(x)| \leq 1$ for all x .

$$\begin{aligned}
p_n &= (\delta \cdot \rho e^{i\varphi})^n + \rho^{2n} + (\delta \cdot \rho e^{-i\varphi})^n \\
&= 2\delta^n \rho^n \cos(n\varphi) + \rho^{2n} \\
&= 2(\sqrt{\delta})^n \cos(n\varphi) + \delta^{-n} \\
&= 2(\sqrt{\delta})^n T_n(\cos \varphi) + (\delta^2 + 1)^n
\end{aligned}$$

where $\cos \varphi = -\delta/(2\rho) = -\delta\sqrt{\delta}/2$. Since the Chebyshev polynomial T_n has parity $(-1)^n$ we get

$$p_n = 2(-\sqrt{\delta})^n T_n(\delta\sqrt{\delta}/2) + (\delta^2 + 1)^n . \quad \square$$

We will now express a_n in p_n and start with D_n . $N = 3$ in (9) gives

$$D_n = \sum_{k=1}^3 \frac{1}{(3 - 2x_k)x_k^n} .$$

At first we simplify the least common denominator,

$$\begin{aligned}
\prod_{k=1}^3 (3 - 2x_k)x_k^n &= (3 - 2x_1)(3 - 2x_2)(3 - 2x_3)(x_1x_2x_3)^n \\
&= 27 - 18(x_3 + x_1 + x_2) + 12(x_1x_3 + x_2x_3 + x_1x_2) - 8x_1x_2x_3 \\
&= 27 - 18 \cdot 0 + 12 \cdot 1 - 8 \cdot 1 = 31
\end{aligned}$$

and then we get

$$\begin{aligned}
D_n &= \frac{1}{31} \{ (3 - 2x_1)x_1^n(3 - 2x_2)x_2^n + (3 - 2x_2)x_2^n(3 - 2x_3)x_3^n + (3 - 2x_3)x_3^n(3 - 2x_1)x_1^n \} \\
&= \frac{1}{31} \{ (9 - 6(x_1 + x_2) + 4x_1x_2)(x_1x_2)^n + (9 - 6(x_2 + x_3) + 4x_2x_3)(x_2x_3)^n \\
&\quad + (9 - 6(x_3 + x_1) + 4x_3x_1)(x_3x_1)^n \} \\
&= \frac{1}{31} (9p_n + 4p_{n+1} - 6q_n)
\end{aligned}$$

where

$$q_n = x_1^{n+1}x_2^n + x_1^nx_2^{n+1} + x_2^{n+1}x_3^n + x_2^nx_3^{n+1} + x_3^{n+1}x_1^n + x_3^nx_1^{n+1} .$$

Next we express q_n in p_n . Since $x_1x_2 + x_2x_3 + x_3x_1 = 1$,

$$\begin{aligned} p_n &= (x_1x_2 + x_2x_3 + x_3x_1) [(x_1x_2)^n + (x_2x_3)^n + (x_3x_1)^n] \\ &= p_{n+1} + q_{n-1} . \end{aligned}$$

Since this holds for all n it can be written

$$q_n = p_{n+1} - p_{n+2}$$

whereupon D_n can be written

$$D_n = \frac{1}{31}(9p_n - 2p_{n+1} + 6p_{n+2})$$

for $N = 3$.

In the case $\alpha_0 = 0$, $\alpha_1 = 1$, $\alpha_2 = 2$ we get, by means of equation (6),

$$\begin{aligned} a_n &= D_{n-1} + D_{n-2} \\ &= \frac{1}{31}(6p_{n+1} + 4p_n + 7p_{n-1} + 9p_{n-2}) . \end{aligned}$$

For example,

$$\begin{aligned} a_3 &= \frac{1}{31}(6p_4 + 4p_3 + 7p_2 + 9p_1) \\ &= \frac{1}{31}\{12\delta^2(8 \cdot \frac{\delta^6}{16} - 8 \cdot \frac{\delta^3}{4} + 1) - 8\delta\sqrt{\delta}(4 \cdot \frac{\delta^4\sqrt{\delta}}{8} - 3 \cdot \frac{\delta\sqrt{\delta}}{2}) + 14\delta(2 \cdot \frac{\delta^3}{4} - 1) \\ &\quad - 18\sqrt{\delta} \cdot \frac{\delta\sqrt{\delta}}{2} + 6(\delta^2 + 1)^4 + 4(\delta^2 + 1)^3 + 7(\delta^2 + 1)^2 + 9(\delta^2 + 1)\} \\ &= \dots = \\ &= \frac{1}{31}\{12\delta^2 - 24(1 - \delta) + 12(\delta - \delta^2) + 24 - 48\delta + 24\delta^2 - 24\delta^2 + 24(1 - \delta) \\ &\quad + 62\delta - 62\delta^2 + 12 - 12\delta + 62\delta^2 - 14\delta + 26\} = \dots = 62/31 = 2 . \end{aligned}$$

Obviously not any particularly efficient method for the calculation of particular terms a_n of a number sequence.

6 Estimate of p_n for $N = 3$ and one more formula for a_n

From the above,

$$p_n = 2(-\sqrt{\delta})^n \cdot T_n\left(\frac{\delta\sqrt{\delta}}{2}\right) + (\delta^2 + 1)^n$$

where δ is the real zero of the polynomial $x^3 + x - 1$,

$$\delta = \sqrt[3]{\frac{\sqrt{93}}{18} + \frac{1}{2}} - \sqrt[3]{\frac{\sqrt{93}}{18} - \frac{1}{2}}$$

and

$$(\sqrt{\delta})^n < 1/8 \text{ if } n > 10 .$$

Since $|T_n(x)| \leq 1$ because $T_n(x)$ is a cosine value, then $|p_n - (\delta^2 + 1)^n| < 1/4$ from which follows that

$$p_n = \lfloor (\delta^2 + 1)^n + 1/2 \rfloor$$

and thus a_n can be written exactly

$$a_n = \frac{1}{31} \{ 6 \lfloor (\delta^2 + 1)^{n+1} + 1/2 \rfloor + 4 \lfloor (\delta^2 + 1)^n + 1/2 \rfloor + 7 \lfloor (\delta^2 + 1)^{n-1} + 1/2 \rfloor + 9 \lfloor (\delta^2 + 1)^{n-2} + 1/2 \rfloor \}$$

for $n > 10$ or, alternatively,

$$a_n = \frac{1}{31} \{ 6 \lfloor \epsilon^{n+1} + 1/2 \rfloor + 4 \lfloor \epsilon^n + 1/2 \rfloor + 7 \lfloor \epsilon^{n-1} + 1/2 \rfloor + 9 \lfloor \epsilon^{n-2} + 1/2 \rfloor \}$$

where

$$\epsilon = \frac{1}{\delta} = \sqrt[3]{\frac{29}{54} + \frac{\sqrt{93}}{18}} + \sqrt[3]{\frac{29}{54} - \frac{\sqrt{93}}{18}} + \frac{1}{3} .$$

Example:

$$\begin{aligned}
a_{16} &= \frac{1}{31} \{6\lfloor \epsilon^{17} + 1/2 \rfloor + 4\lfloor \epsilon^{16} + 1/2 \rfloor + 7\lfloor \epsilon^{15} + 1/2 \rfloor + 9\lfloor \epsilon^{14} + 1/2 \rfloor\} \\
&= \frac{1}{31} (6 \cdot 664 + 4 \cdot 453 + 7 \cdot 309 + 9 \cdot 211) = 318 .
\end{aligned}$$

Corresponding formula for the sequence $(\kappa_n)_{n=0}^\infty$:

$$\begin{aligned}
\kappa_n &= D_{n-1} = \frac{1}{31} (9p_{n-1} - 2p_n + 6p_{n+1}) \\
&= \frac{1}{31} \{6\lfloor \epsilon^{n+1} + 1/2 \rfloor - 2\lfloor \epsilon^n + 1/2 \rfloor + 9\lfloor \epsilon^{n-1} + 1/2 \rfloor\} .
\end{aligned}$$

Similarly, but essentially simpler, the Fibonacci numbers can be expressed by means of the floor function $\lfloor \cdot \rfloor$:

$$F_n = \left\lfloor \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2} \right)^n + \frac{1}{2} \right\rfloor$$

(for all n).

Also see [1], p 300.

7 General linear recursion relation

$a_{n+N} = \sum_{u=1}^N A_u a_{n+N-u}$ and the initial values $(\alpha_n)_{n=0}^{N-1}$.

Generating function:

$$\begin{aligned}
f(x) &= \sum_{n=0}^{\infty} a_n x^n = x^N \sum_{n=0}^{\infty} a_{n+N} x^n + \sum_{n=0}^{N-1} \alpha_n x^n \\
&= x^N \sum_{n=0}^{\infty} \sum_{u=1}^N A_u a_{n+N-u} x^n + \sum_{n=0}^{N-1} \alpha_n x^n \\
&= \sum_{u=1}^{N-1} A_u x^u \sum_{n=0}^{\infty} a_{n+N-u} x^{n+N-u} + A_N x^N \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{N-1} \alpha_n x^n \\
&= \sum_{u=1}^N A_u x^u \cdot f(x) - \sum_{u=1}^{N-1} \sum_{n=0}^{N-u-1} A_u \alpha_n x^{u+n} + \sum_{n=0}^{N-1} a_n x^n
\end{aligned}$$

giving

$$\left(1 - \sum_{n=1}^N A_n x^n\right) \cdot f(x) = \sum_{n=0}^{N-1} \alpha_n x^n - \sum_{u=1}^{N-1} \sum_{n=0}^{N-u-1} A_n \alpha_n x^{u+n} .$$

Example: $N = 3$

$$\sum_{u=1}^{N-1} \sum_{n=0}^{N-u-1} A_u \alpha_n x^{u+n} = A_1 \alpha_0 x + A_1 \alpha_1 x^2 + A_2 \alpha_0 x^2$$

from which

$$f(x) = \frac{x^2(\alpha_2 - A_1 \alpha_1 - A_2 \alpha_0) + x(\alpha_1 - A_1 \alpha_0) + \alpha_0}{1 - \sum_{n=1}^3 A_n x^n} .$$

8 Conclusion

Given a relation $a_n = a_{n-1} + a_{n-N}$ and N initial values you can, obviously, calculate any value using the recursion relation repeatedly. But a closed formula would directly give the desired value. This, however, proved rather intricate when $N > 2$ since the zeroes of a polynomial of degree N will occur in the formula. A formula derived for $N = 3$ is relatively simple but yet tedious to use for large index values. Using, however, a suitable computer program, such as DERIVE, it will work well. Apart from this a closed formula is, of course, interesting in itself. Further, a method has been deduced making it possible to calculate any desired value a_n by associating the value to another number sequence $(\kappa_n)_{n=0}^{\infty}$, which is easier to calculate. This method, which I regard as the most interesting thing of this article, is an inbetween of directly iterating the recursion formula and using an expression in closed form.

Another method of finding an explicit formula for the terms in a number sequence, besides using a generating function, is to determine the zeroes of the characteristic polynomial for the recursion relation, analogously with a method of solving linear, ordinary differential equations. For $N = 2$ the amount of work as well as the formula obtained will be reasonably equal, but for $N = 3$ essentially worse as far as I have investigated. It would be of some interest to compare the methods in general. The algorithm using the sequence (κ) works principally the same way independently of N . Finally, it might be interesting to investigate the influence of different N on the zeroes

of the polynomial $x^N + x - 1$ or the characteristic polynomial $x^N - x^{N-1} - 1$, which also has degree N .⁶

Appendix

The initial values for κ

Proof that $\kappa_n = 1$ for $0 < n < N$ ($N \geq 2$).

Notation: $\kappa_n \equiv \kappa_{n,N-1}^N$.

$$\begin{aligned}
 (\delta^{N-1} + 1)^n &= \sum_{\nu=0}^n \binom{n}{\nu} \delta^{\nu N - \nu} \\
 &= \sum_{\nu=0}^n \binom{n}{\nu} (1 - \delta)^{\nu-1} \delta^{N-\nu} \\
 &= 1 + \sum_{\nu=1}^n \sum_{\mu=0}^{\nu-1} \binom{n}{\nu} \binom{\nu-1}{\mu} (-1)^\mu \delta^{N-\nu+\mu}
 \end{aligned}$$

where

$$1 \leq N - n \leq N - \nu \leq N - \nu + \mu \leq N - \nu + \nu - 1 = N - 1$$

whence

$$1 \leq N - \nu + \mu \leq N - 1 .$$

The coefficient of δ^{N-1} , i e $\kappa_n = \kappa_{n,N-1}^N$ is obtained for $\mu - \nu = -1$, i e

$$\begin{aligned}
 \kappa_n &= \sum_{\nu=1}^n \binom{n}{\nu} \binom{\nu-1}{\nu-1} (-1)^{\nu-1} \\
 &= - \sum_{\nu=0}^n \binom{n}{\nu} (-1)^\nu + \binom{n}{0} (-1)^0 \\
 &= -(1-1)^n + 1 = 1 .
 \end{aligned}$$

⁶These polynomial have a simple connection: If one polynomial has zeroes x_k , then the other has zeroes $1/x_k$. They are essentially reciprocal polynomials of each other.

References

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- [3] <http://math.stackexchange.com/questions/393646/irreducibility-of-xn-x-1-over-mathbb-q>